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Existence and uniqueness of solution for two one-phase Stefan problems with variable thermal coefficients

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ABSTRACT

One dimensional Stefan problems for a semi-infinite material with temperature dependent thermal coefficients are considered. Existence and uniqueness of solution are obtained imposing a Dirichlet, a Neumann or a Robin type condition at fixed face $x = 0$. Moreover, it is proved that the solution of the problem with the Robin type condition converges to the solution of the problem with the Dirichlet condition at the fixed face. Computational examples are provided.

1. Introduction

The one-phase Stefan problem (or Lamé–Clapeyron–Stefan problem) for a semi-infinite material is a free boundary problem for the heat equation, which requires the determination of the temperature distribution T of the liquid phase (melting problem) or the solid phase (solidification problem) and the evolution of the free boundary $x = s(t)$. Phase change problems appear frequently in industrial processes and other problems of technological interest [1–6]. The Lamé–Clapeyron–Stefan problem is non-linear even in its simplest form due to the free boundary conditions. If the thermal coefficients of the material are temperature-dependent, we have a doubly non-linear free boundary problem. Some other models involving temperature-dependent thermal conductivity can also be found in [7–21] and with variable latent heat in [22,23].

In this paper, we consider two one-phase fusion problems with a temperature-dependent thermal conductivity $k(T)$ and specific heat $c(T)$. In one of them, it is assumed a Dirichlet condition at the fixed face $x = 0$ and in the second case a Robin condition is imposed. The mathematical model of the governing process is described as follows:

$$\rho c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right), \qquad 0 < x < s(t), \quad t > 0, \tag{1.1}$$

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$$T(0,t) = T_0, \qquad t > 0, \tag{1.2}$$

$$T(s(t),t) = T_f, \qquad t > 0, \tag{1.3}$$

$$k_0 \frac{\partial T}{\partial x}(s(t),t) = -\rho l \dot{s}(t), \qquad t > 0, \tag{1.4}$$

$$s(0) = 0, \tag{1.5}$$

where the unknown functions are the temperature $T = T(x,t)$ and the free boundary $x = s(t)$ separating both phases. The parameters $\rho > 0$ (density), $l > 0$ (latent heat per unit mass), $T_0 > 0$ (temperature imposed at the fixed face $x = 0$) and $T_f < T_0$ (phase change temperature at the free boundary $x = s(t)$) are all known constants. The functions k and c are defined as:

$$k(T) = k_0 \left(1 + \delta \left(\frac{T-T_f}{T_0-T_f}\right)^p\right) \tag{1.6}$$

$$c(T) = c_0 \left(1 + \delta \left(\frac{T-T_f}{T_0-T_f}\right)^p\right), \tag{1.7}$$

where δ and p are given non-negative constants, $k_0 = k(T_f)$ and $c_0 = c(T_f)$ are the reference thermal conductivity and the specific heat, respectively.

The problem (1.1)–(1.5) was firstly considered in [24] where an equivalent ordinary differential problem was obtained. In [25], the existence of an explicit solution of a similarity type by using a double fixed point was given when the thermal coefficients are bounded and Lipschitz functions.

We are interested in obtaining a similarity solution to problem (1.1)–(1.5). More precisely, one in which the temperature $T = T(x,t)$ can be written as a function of a single variable. Through the following change of variables:

$$y(\eta) = \frac{T(x,t)-T_f}{T_0-T_f} \geq 0 \tag{1.8}$$

with

$$\eta = \frac{x}{2a\sqrt{t}}, \quad 0 < x < s(t), \quad t > 0, \tag{1.9}$$

the phase front moves as

$$s(t) = 2a\lambda\sqrt{t} \tag{1.10}$$

where $a^2 = \frac{k_0}{\rho c_0}$ (thermal diffusivity) and $\lambda > 0$ is a positive parameter to be determined.

It is easy to see that the Stefan problem (1.1)–(1.5) has a similarity solution (T, s) given by:

$$T(x,t) = (T_0 - T_f)\, y\left(\frac{x}{2a\sqrt{t}}\right) + T_f, \quad 0 < x < s(t), \quad t > 0, \tag{1.11}$$

$$s(t) = 2a\lambda\sqrt{t}, \qquad t > 0 \tag{1.12}$$

if and only if the function y and the parameter $\lambda > 0$ satisfy the following ordinary differential problem:

$$2\eta(1 + \delta y^p(\eta))y'(\eta) + [(1 + \delta y^p(\eta))y'(\eta)]' = 0, \qquad 0 < \eta < \lambda, \tag{1.13}$$

$$y(0) = 1, \tag{1.14}$$

$$y(\lambda) = 0, \tag{1.15}$$

$$y'(\lambda) = -\frac{2\lambda}{\mathrm{Ste}} \tag{1.16}$$

where $\delta \geq 0$, $p \geq 0$ and $\mathrm{Ste} = \frac{c_0(T_0-T_f)}{l} > 0$ is the Stefan number.

In [24], the solution to the ordinary differential problem (1.13)–(1.16) was approximated by using shifted Chebyshev polynomials. Although, in this paper was provided the exact solution for the particular cases $p = 1$ and $p = 2$, the aim of our work is to prove existence and uniqueness of solution for every $\delta \geq 0$ and $p \geq 0$. The particular case with $\delta = 0$, i.e. with constant thermal coefficients, and $p = 1$ was studied in [13,14,26,27].

In Section 2, we are going to prove existence and uniqueness of problem (1.1)–(1.5) through analysing the ordinary differential problem (1.13)–(1.16).

In Section 3, we will present a similar problem but with a generalized condition at the fixed face $x = 0$ as in [28]:

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}\left(\varepsilon T(0,t) - T_0\right), \tag{1.17}$$

with $\varepsilon \in [0,1]$.

If we specify $\varepsilon = 0$, we can rewrite (1.17) as a Neumann condition:

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = -\frac{q}{\sqrt{t}}, \tag{1.18}$$

where $q = hT_0$ characterizes prescribed heat flux at the fixed face $x = 0$.

For the special case $\varepsilon = 1$, the condition (1.17) represents a Robin type condition given by

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}\left(T(0,t) - T_0\right), \tag{1.19}$$

where $h > 0$ is the generalized thermal transfer coefficient and T_0 is the bulk temperature.

We will prove existence and uniqueness of solution to both problems, in a similar way that we did for the preceding section.

Finally, in Section 4, we will study the asymptotic behaviour when $h \to +\infty$, for the problem with Robin type condition (1.19).

2. Existence and uniqueness of solution to the problem with Dirichlet condition at the fixed face $x = 0$

We will study the existence and uniqueness of solution to the problem (1.1)–(1.5) through the ordinary differential problem (1.13)–(1.16).

Lemma 2.1. *Let $p \geq 0$, $\delta \geq 0$, $\lambda > 0$, $y \in C^{\infty}[0,\lambda]$ and $y \geq 0$, then (y,λ) is a solution to the ordinary differential problem (1.13)–(1.16) if and only if λ is the unique solution to*

$$f(x) = g, \qquad x > 0, \tag{2.1}$$

and y verifies

$$F(y(\eta)) = G(\eta), \qquad 0 < \eta < \lambda, \tag{2.2}$$

where

$$g = 1 + \tfrac{\delta}{p+1}, \qquad f(x) = \tfrac{\sqrt{\pi}}{\text{Ste}} x \exp(x^2)\operatorname{erf}(x), \tag{2.3}$$

$$F(x) = x + \tfrac{\delta}{p+1} x^{p+1}, \qquad G(x) = \tfrac{\sqrt{\pi}}{\text{Ste}}\,\lambda \exp(\lambda^2)\left(\operatorname{erf}(\lambda) - \operatorname{erf}(x)\right). \tag{2.4}$$

Proof. Let (y,λ) be a solution to problem (1.13)–(1.16).

Let us define $v(\eta) = (1 + \delta y^p(\eta))\, y'(\eta)$. Taking into account the ordinary differential equation (1.13) and condition (1.14), v can be rewritten as $v(\eta) = (1+\delta) y'(0)\exp(-\eta^2)$. Therefore

$$y'(\eta) + \delta y^p(\eta) y'(\eta) = (1+\delta) y'(0)\exp(-\eta^2). \tag{2.5}$$

If we integrate (2.5) from 0 to η, and using conditions (1.14)–(1.15) we obtain

$$y(\eta) + \tfrac{\delta}{p+1} y^{p+1}(\eta) = 1 + \tfrac{\delta}{p+1} - \tfrac{\sqrt{\pi}}{\text{Ste}}\lambda \exp(\lambda^2)\operatorname{erf}(\eta) \tag{2.6}$$

If we take $\eta = \lambda$ in the above equation, by (1.15), we get (2.1). Furthermore, from (2.1) we can rewrite (2.6) as (2.2).

Reciprocally, if (y, λ) is a solution to (2.1)–(2.2) we have

$$y(\eta) = -\tfrac{\delta}{p+1} y^{p+1}(\eta) + \left(1 + \tfrac{\delta}{p+1}\right)\left(1 - \tfrac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)}\right). \tag{2.7}$$

An easy computation shows that (y, λ) is a solution to the ordinary differential problem (1.13)–(1.16). □

According to the above result, we proceed to show that there exists a unique solution to problem (2.1)–(2.2).

Lemma 2.2. *If $p \geq 0$ and $\delta \geq 0$, then there exists a unique solution (y, λ) to the problem (2.1)–(2.2) with $\lambda > 0$, $y \in C^{\infty}[0, \lambda]$ and $y \geq 0$.*

Proof. In virtue that f given by (2.3) is an increasing function such that $f(0) = 0$ and $f(+\infty) = +\infty$, there exists a unique solution $\lambda > 0$ to Eq. (2.1). Now, for this $\lambda > 0$, it is easy to see that F given by (2.4) is an increasing function, so that we can define $F^{-1} : [0, +\infty) \to [0, +\infty)$. As G defined by (2.4) is a positive function, we have that there exists a unique solution $y \in C^{\infty}[0, \lambda]$ of Eq. (2.2) given by

$$y(\eta) = F^{-1}(G(\eta)), \qquad 0 < \eta < \lambda. \quad \square \tag{2.8}$$

Remark 2.3. On one hand we have that F is an increasing function with $F(0) = 0$ and $F(1) = 1 + \frac{\delta}{p+1}$. On the other hand, G is a decreasing function with $G(0) = 1 + \frac{\delta}{p+1}$ and $G(\lambda) = 0$. Then it follows that $0 \leq y(\eta) \leq 1$, for $0 < \eta < \lambda$.

From the above lemmas we are able to claim the following result:

Theorem 2.4. *The Stefan problem governed by (1.1)–(1.5) has a unique similarity type solution given by (1.11)–(1.12) where (y, λ) is the unique solution to the functional problem (2.1)–(2.2).*

Remark 2.5. In virtue of Remark 2.3 and Theorem 2.4 we have that

$$T_f < T(x,t) < T_0, \qquad 0 < x < s(t), \quad t > 0.$$

Remark 2.6. If T is a solution of the free boundary problem (1.1)–(1.5) we can define the Kirchhoff transformation

$$\theta(x,t) = \int_{T_f}^{T(x,t)} \left[1 + \delta \left(\tfrac{\xi - T_f}{T_0 - T_f}\right)^{p}\right] \mathrm{d}\xi \tag{2.9}$$

and we obtain for the new unknown θ the classical one-phase Stefan problem with constant thermal coefficient given by

$$\frac{\partial \theta}{\partial t} = \alpha_0 \frac{\partial \theta^2}{\partial x^2}, \qquad 0 < x < s(t), \quad t > 0, \tag{2.10}$$

$$\theta(0,t) = (T_0 - T_f)\left(1 + \tfrac{\delta}{1+p}\right), \qquad t > 0, \tag{2.11}$$

$$\theta(s(t),t) = 0, \qquad t > 0, \tag{2.12}$$

$$k_0 \frac{\partial \theta}{\partial x}(s(t),t) = -\rho l \dot{s}(t), \qquad t > 0, \tag{2.13}$$

$$s(0) = 0, \tag{2.14}$$

whose solution is given by: [1]

$$\begin{cases} \theta(x,t) = (T_0 - T_f)\left(1 + \frac{\delta}{p+1}\right)\left[1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_0 t}}\right)}{\operatorname{erf}(\lambda)}\right] \\ s(t) = 2\lambda\sqrt{\alpha_0 t} \end{cases}$$

where $\lambda > 0$ is the unique solution to the equation:

$$x \operatorname{erf}(x) \exp(x^2) = \frac{\text{Ste}}{\sqrt{\pi}}\left(1 + \frac{\delta}{2}\right), \quad x > 0 \tag{2.15}$$

with

$$\text{Ste} = \frac{c_0\,(T_0 - T_f)}{l}. \tag{2.16}$$

Conversely, if θ is the solution of the free boundary (2.10)–(2.14) then the temperature $T = T(x,t)$ defined by considering

$$\frac{\partial T}{\partial x} = \frac{k_0}{k(T)}\frac{\partial \theta}{\partial x}, \qquad \frac{\partial T}{\partial t} = \frac{k_0}{k(T)}\frac{\partial \theta}{\partial t} \tag{2.17}$$

and the equivalent expression of (2.9) given by

$$\theta(x,t) = (T(x,t) - T_f)\left[\left(1 + \frac{\delta}{p+1}\left(\frac{T(x,t)-T_f}{T_0 - T_f}\right)^p\right)\right] \tag{2.18}$$

is a solution of the problem (1.1)–(1.5). In any case, the explicit solution of the free boundary problem (1.1)–(1.5) is given by the expressions (1.11)–(1.12) as it was proved in Theorem 2.4.

Remark 2.7. For the particular case $p = 1$, $\delta \geq 0$, the solution to the problem (2.1)–(2.2) is given by

$$y(\eta) = \frac{1}{\delta}\left[\sqrt{(1+\delta)^2 - \delta(2+\delta)\frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)}} - 1\right], \qquad 0 < \eta < \lambda, \tag{2.19}$$

where λ verifies

$$\lambda \exp(\lambda^2)\operatorname{erf}(\lambda) = \frac{\text{Ste}}{\sqrt{\pi}}\left(1 + \frac{\delta}{2}\right). \tag{2.20}$$

In fact, if $p = 1$ Eq. (2.2) is given by

$$y^2(\eta) + \frac{2}{\gamma}y(\eta) - (1 + \frac{2}{\gamma})\left[1 - \frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)}\right] = 0 \tag{2.21}$$

which has a unique positive solution obtained by the expression (2.19).

In view of Lemma 2.2 and Remark 2.3, we can compute the solution (y, λ) to the ordinary differential problem (1.13)–(1.16), by using its functional formulation.

In Fig. 1, for different values of p, we plot the solution (y, λ) to the problem (2.1)–(2.2). In order to compare the obtained solution y, we extend them by zero for every $\eta > \lambda$. We assume $\delta = 5$ and Ste $= 0.5$. It must be pointed out that the choice for Ste is due to the fact that for most phase-change material candidates over a realistic temperature, the Stefan number will not exceed 1 (see [29]).

Although it can be analytically deduced from Eq. (2.1), we can observe graphically that as p increases, the value of λ decreases.

In view of Lemma 2.1, we can also plot the solution (T, s) to the problem (1.13)–(1.16).

In Fig. 2 we present a colourmap for the temperature $T = T(x,t)$ extending it by zero for $x > s(t)$.

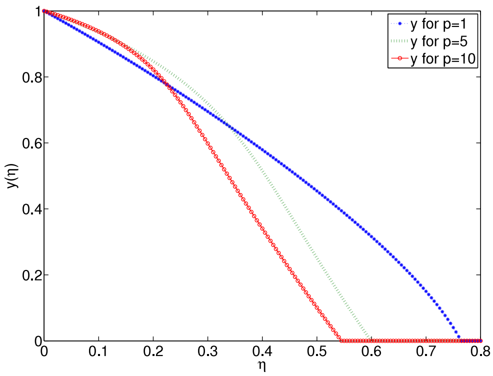

Fig. 1. Plot of function y for different values of $p = 1, 5, 10$, fixing $\delta = 5$ and Ste $= 0.5$.

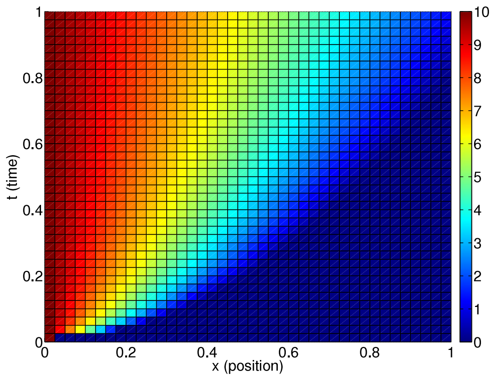

Fig. 2. Colourmap for the temperature $T = T(x,t)$ function fixing $\delta = 1$, $p = 1$, Ste $= 0.5$, $T_f = 0$, $T_0 = 10$ and $a = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3. Existence and uniqueness of solution to the problem with Neumann and Robin conditions at the fixed face $x = 0$

In this section we are going to consider a Stefan problem with a generalized boundary condition at the fixed face (1.17) that represents a Neumann or a Robin condition for the cases $\varepsilon = 0$, $\varepsilon = 1$, respectively. This heat input is the true relevant physical condition due to the fact that it establishes that the incoming flux at the fixed face is proportional to the difference between the temperature at the surface of the material and the ambient temperature to be imposed.

Let us consider the free boundary problem given by (1.1), (1.3)–(1.5) and the convective condition (1.17) instead of the temperature condition (1.2) at the fixed face $x = 0$.

The temperature-dependent thermal conductivity $k(T)$ and the specific heat $c(T)$ are given by (1.6) and (1.7), respectively.

As in the above section, we are searching a similarity type solution. If we define the change of variables as (1.8)–(1.9), the phase front moves as (1.10) where $a^2 = \frac{k_0}{\rho c_0}$ (thermal diffusivity) and λ_γ is a positive parameter to be determined.

It follows that (T_γ, s_γ) is a solution to (1.1), (1.3)–(1.5) and (1.17) if and only if the function y_γ defined by (1.13) and the parameter $\lambda_\gamma > 0$ given by (1.10) satisfy (1.13), (1.15), (1.16) and

$$(1 + \delta y^p(0))\, y'(0) = \gamma\, (\varepsilon y(0) - 1)\,, \tag{3.1}$$

where $\delta \geq 0$, $p \geq 0$, $\varepsilon \in [0,1]$,

$$\gamma = 2\text{Bi} \quad \text{and} \quad \text{Bi} = \frac{ha}{k_0}, \tag{3.2}$$

where $\text{Bi} > 0$ is the generalized Biot number.

With a few slight changes on the results obtained in the previous section, the following assertions can be established:

Lemma 3.1. *Let $p \geq 0$, $\delta \geq 0$, $\gamma > 0$, $\varepsilon \in [0,1]$, $\lambda_\gamma > 0$, $y_\gamma \in C^\infty[0,\lambda_\gamma]$ and $y_\gamma \geq 0$, then $(y_\gamma, \lambda_\gamma)$ is a solution to the ordinary differential problem (1.13), (1.15), (1.16) and (3.1) if and only if λ_γ is the unique solution to the following equation*

$$F_\varepsilon(\beta_\gamma(x)) = f_\varepsilon(x), \qquad x > 0, \tag{3.3}$$

and y_γ verifies

$$F(y_\gamma(\eta)) = G_\gamma(\eta), \qquad 0 < \eta < \lambda_\gamma \tag{3.4}$$

where F is given by (2.4) and

$$F_\varepsilon(x) = \varepsilon^p x + \tfrac{\delta}{p+1} x^{p+1}, \tag{3.5}$$

$$f_\varepsilon(x) = \varepsilon^{p+1} \tfrac{\sqrt{\pi}}{\text{Ste}} x \exp(x^2)\,\text{erf}(x), \tag{3.6}$$

$$\beta_\gamma(x) = 1 - \tfrac{2x \exp(x^2)}{\gamma\, Ste}, \qquad 0 \leq x \leq \lambda_0 = \beta_\gamma^{-1}(0), \tag{3.7}$$

$$G_\gamma(x) = \frac{\lambda_\gamma \exp\left(\lambda_\gamma^2\right)\sqrt{\pi}}{Ste}\left(\text{erf}(\lambda_\gamma) - \text{erf}(x)\right), \quad 0 < x < \lambda_\gamma. \tag{3.8}$$

Proof. Let $(y_\gamma, \lambda_\gamma)$ be a solution to problem (1.13), (1.15), (1.16) and (3.1).

Let us define $w(\eta) = \left(1 + \delta y_\gamma^p(\eta)\right) y_\gamma'(\eta)$. Taking into account the ordinary differential equation (1.13) and the conditions (1.15), (3.1), w can be rewritten as $w(\eta) = y_\gamma'(\lambda_\gamma)\exp(\lambda_\gamma^2)\exp(-\eta^2)$. Therefore

$$y_\gamma'(\eta) + \delta y_\gamma^p(\eta) y_\gamma'(\eta) = y_\gamma'(\lambda_\gamma)\exp(\lambda_\gamma^2)\exp(-\eta^2). \tag{3.9}$$

If we integrate (2.5) from η to λ_γ and using conditions (1.15), (1.16) and (3.1) we obtain that y_γ verifies (3.4).

If we take $\eta = 0$ in (3.4) we get

$$y_\gamma(0) + \tfrac{\delta}{p+1} y_\gamma^{p+1}(0) = \tfrac{\sqrt{\pi}}{\text{Ste}} \lambda_\gamma \exp(\lambda_\gamma^2)\,\text{erf}(\lambda_\gamma). \tag{3.10}$$

Furthermore, if we differentiate equation (3.4) and computing this derivative at $\eta = 0$ we obtain:

$$y_\gamma'(0) + \delta y_\gamma^p(0) y_\gamma'(0) = -\tfrac{2\lambda_\gamma \exp(\lambda_\gamma^2)}{\text{Ste}}, \tag{3.11}$$

and from (3.1) and (3.11) we obtain that (3.3) holds.

Reciprocally, if $(y_\gamma, \lambda_\gamma)$ is a solution to (3.3)–(3.4), an easy computation shows that $(y_\gamma, \lambda_\gamma)$ verifies (1.13), (1.15), (1.16) and (3.1). □

Remark 3.2. The notations λ_γ and y_γ are adopted in order to emphasize the dependence of the solution to problem (1.13), (1.15), (1.16) and (3.1) on γ, although it also depends on p and δ. This fact is going to facilitate the subsequent analysis of the asymptotic behaviour of y_γ when $\gamma \to \infty$ ($h \to \infty$) to be presented in Section 4.

Lemma 3.3. *If $p \geq 0$, $\delta \geq 0$, $\gamma > 0$ and $\varepsilon \in [0,1]$, then there exists a unique solution $(y_\gamma, \lambda_\gamma)$ to the problem (3.3)–(3.4) with $\lambda_\gamma > 0$, $y_\gamma \in C^\infty[0,\lambda_\gamma]$ and $y_\gamma \geq 0$.*

Proof. On one hand, the function f_ε given by (3.6) is an increasing function such that $f_\varepsilon(0) = 0$ and $f_\varepsilon(\lambda_0) > 0$ with $\lambda_0 = \beta_\gamma^{-1}(0)$. On the other hand, $F_\varepsilon(\beta_\gamma)$ with F_ε given by (3.5) and β_γ given by (3.7), is a decreasing function for $0 \leq x \leq \lambda_0$. Notice that $F_\varepsilon(\beta_\gamma(0)) = F_\varepsilon(1) = \varepsilon^p + \frac{\delta}{p+1}$ and $F_\varepsilon(\beta_\gamma(\lambda_0)) = F_\varepsilon(0) = 0$. Therefore we can conclude that there exists a unique $0 < \lambda_\gamma < \lambda_0$ that verifies (3.3).

Now, for this $\lambda_\gamma > 0$, it is easy to see that F is an increasing function, so that we can define $F^{-1} : [0, +\infty) \to [0, +\infty)$. As G_γ given by (3.8) is a positive function, we have that there exists a unique solution $y \in C^\infty[0, \lambda_\gamma]$ of Eq. (3.4) given by

$$y_\gamma(\eta) = F^{-1}\left(G_\gamma(\eta)\right), \qquad 0 < \eta < \lambda_\gamma. \quad \square \tag{3.12}$$

Remark 3.4. On one hand we have that F is an increasing function with $F(0) = 0$ and $F(1) = 1 + \frac{\delta}{p+1}$. On the other hand, G_γ is a decreasing function with $G_\gamma(0) = \lambda_\gamma \exp(\lambda_\gamma^2)\,\mathrm{erf}(\lambda_\gamma)$ and $G_\gamma(\lambda_\gamma) = 0$. Then y_γ is a decreasing function and due to (3.3) we obtain

$$y_\gamma(0) = F^{-1}(G_\gamma(0)) = \beta_\gamma(\lambda_\gamma) = 1 - \frac{2\lambda_\gamma \exp(\lambda_\gamma^2)}{\gamma\,\mathrm{Ste}} < 1.$$

Then it follows that $0 \leq y_\gamma(\eta) \leq 1$ for $0 < \eta < \lambda_\gamma$.

Finally, from the above lemmas we are able to claim the following result:

Theorem 3.5. *The Stefan problem governed by (1.1), (1.3)–(1.5) and (1.17) has a unique similarity type solution given by (1.11)–(1.12) where $(y_\gamma, \lambda_\gamma)$ is the unique solution to the functional problem (3.3)–(3.4).*

The solutions to problems with Neumann or Robin boundary condition at the fixed face can be obtained as a direct consequence of Theorem 3.5 by fixing $\varepsilon = 0$ or $\varepsilon = 1$, respectively. Therefore we have the following results:

Corollary 3.6 (*Case* $\varepsilon = 0$). *The Stefan problem governed by (1.1), (1.3)–(1.5) and the Neumann condition (1.18) has a unique similarity type solution given by (1.11)–(1.12) where y_γ is the unique solution to Eq. (3.3) and λ_γ is the unique solution of the equation:*

$$\frac{2x \exp\left(x^2\right)}{\gamma \mathrm{Ste}} = 1, \quad x > 0.$$

Corollary 3.7 (*Case* $\varepsilon = 1$). *The Stefan problem governed by (1.1), (1.3)–(1.5) and the Robin type condition (1.19) has a unique similarity type solution given by (1.11)–(1.12) where y_γ is the unique solution to Eq. (3.3) and λ_γ is the unique solution of the equation*

$$F\left(\beta_\gamma(x)\right) = f(x), \quad x > 0,$$

with F, f and β_γ given by (2.4), (2.3) and *(3.7).*

Taking into account Lemmas 3.1 and 3.3 we compute the solution $(y_\gamma, \lambda_\gamma)$ to the ordinary differential problem (1.13), (1.15), (1.16) and (3.1), using its functional formulation (3.3)–(3.4). Fig. 3 shows the function y_γ for a fixed $\delta = 5$, $\gamma = 50$, $\varepsilon = 1$, $\mathrm{Ste} = 0.5$, varying $p = 1, 5, 10$. As it was made for the problem with a Dirichlet condition at the fixed face, the solution y_γ is extended by zero for every $\eta > \lambda_\gamma$.

Applying Lemma 3.1, we can also plot the solution (T_γ, s_γ) to the problem (1.1), (1.3)–(1.5) and (1.17). In Fig. 4 we present a colourmap for the temperature $T_\gamma = T_\gamma(x, t)$ extending it by zero for $x > s_\gamma(t)$.

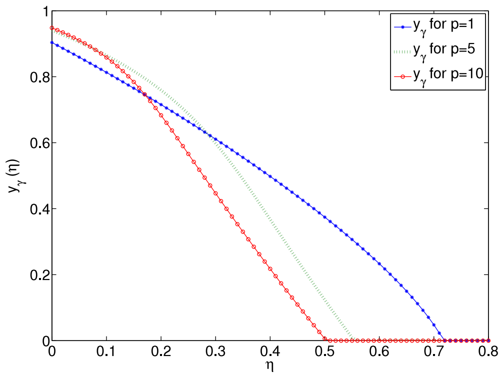

Fig. 3. Plot of function y for different values of $p = 1, 5, 10$, fixing $\delta = 5$, $\gamma = 50$, $\varepsilon = 1$ and Ste $= 0.5$.

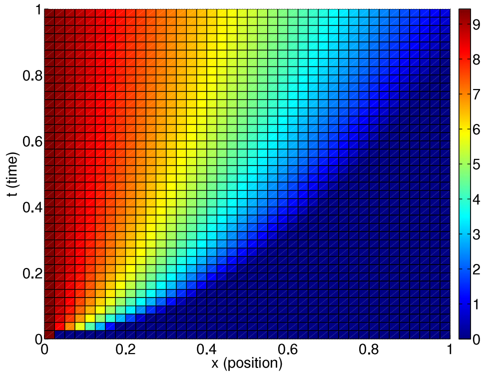

Fig. 4. Colourmap for the temperature $T = T(x, t)$ function fixing $\delta = 1$, $\gamma = 50$, $\varepsilon = 1$, $p = 1$, Ste $= 0.5$, $T_f = 0$, $T_0 = 10$ and $a = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

4. Asymptotic behaviour

Now, we will show that if the coefficient γ, that characterizes the heat transfer at the fixed face, goes to infinity then the solution to the problem with the Robin type condition (1.1), (1.3)–(1.5) and (1.19) converges to the solution to the problem (1.1)–(1.5), with a Dirichlet condition at the fixed face $x = 0$.

In order to get the convergence it will be necessary to prove the following preliminary result:

Lemma 4.1. *Let $\gamma > 0$, $p \geq 0$ and $\delta > 0$ be. If λ_γ is the unique solution to Eq.* (3.3) *and λ is the unique solution to Eq.* (2.1)*, then the sequence $\{\lambda_\gamma\}$ is increasing and bounded. Moreover,*

$$\lim_{\gamma\to\infty} \lambda_\gamma = \lambda.$$

Proof. Let $\gamma_1 < \gamma_2$ then $F(\beta_{\gamma_1}) < F(\beta_{\gamma_2})$ where F is given by (2.4) and β_γ is defined by (3.7). Therefore $\lambda_{\gamma_1} < \lambda_{\gamma_2}$. In addition as $\lim_{\gamma\to\infty} F(\beta_\gamma) = g$ we have $\lambda_\gamma < \lambda$, for all $\gamma > 0$. Finally, we obtain that $\lim_{\gamma\to\infty} \lambda_\gamma = \lambda$. □

Lemma 4.2. *Let $\gamma > 0$, $p \geq 0$ and $\delta > 0$ be. If $(y_\gamma, \lambda_\gamma)$ is the unique solution to the ordinary differential problem* (1.13), (1.15), (1.16), (3.1) *and (y, λ) is the unique solution to the problem* (1.13)–(1.16)*, then for every $\eta \in (0, \lambda)$ the following convergence holds*

$$\lim_{\gamma\to\infty} y_\gamma(\eta) = y(\eta). \tag{4.1}$$

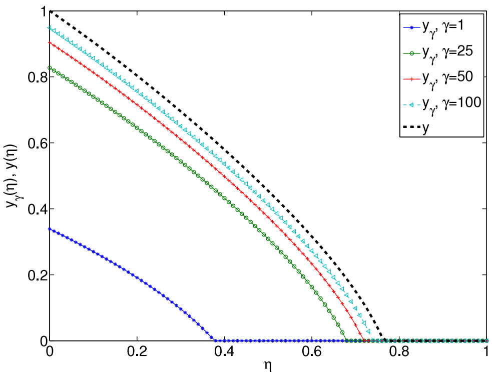

Fig. 5. Plot of y_γ for $\gamma = 1, 25, 50, 100$, and y functions fixing $p = 1$ and $\delta = 5$.

Proof. According to Lemmas 2.2 and 3.3 we have that $y_\gamma(\eta) = F^{-1}(G_\gamma(\eta))$, with $0 < \eta < \lambda_\gamma$ and $y(\eta) = F^{-1}(G(\eta))$, with $0 < \eta < \lambda$ where the functions F, G and G_γ are given by (2.4) and (3.8).

Let $\eta \in (0, \lambda)$. Then due to Lemma 4.2, there exists γ_0 such that $\eta < \lambda_\gamma$, for every $\gamma > \gamma_0$. As it can be easily seen that $G_\gamma(\eta) \to G(\eta)$ when $\gamma \to \infty$, it follows that

$$\lim_{\gamma\to\infty} y_\gamma(\eta) = \lim_{\gamma\to\infty} F^{-1}(G_\gamma(\eta)) = F^{-1}\left(\lim_{\gamma\to\infty} G_\gamma(\eta)\right) = F^{-1}(G(\eta)) = y(\eta). \quad \square$$

In order to illustrate the results obtained in Lemmas 4.1 and 4.2, in Fig. 5 we plot the $(y_\gamma, \lambda_\gamma)$ assuming $\delta = 5$, $p = 1$ and varying $\gamma = 1, 25, 50, 100$. We show that as γ becomes greater, the function y_γ converges pointwise to the solution y of the problem (1.13)–(1.16).

Theorem 4.3. *The unique solution (T_γ, s_γ) to the Stefan problem governed by (1.1), (1.3)–(1.5) and (1.19) converges pointwise to the unique solution (T, s) to the Stefan problem (1.1)–(1.5) when $\gamma \to \infty$.*

Proof. The proof follows straightforward from Lemmas 4.1, 4.2 and formulas (1.11)–(1.12). $\square$

5. Conclusions

One dimensional Stefan problems with temperature dependent thermal coefficients and a Dirichlet, a Neumann or a Robin type condition at fixed face $x = 0$ for a semi-infinite material were considered. Existence and uniqueness of solution was obtained in all cases. Moreover, it was proved that the solution of the problem with the Robin type condition converges to the solution of the problem with the Dirichlet condition at the fixed face. For a particular case, an explicit solution was also obtained. In addition, computational examples were provided in order to show the previous theoretical results.

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